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Solution of Nonlinear Elliptic Equations with Boundary Singularities by an Integral Equation Method

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A boundary integral equation (BIE) formulation is presented for the numerical solution of certain two-dimensional nonlinear elliptic equations subject to nonlinear boundary conditions. By applying the Kirchoff transformation, all nonlinear aspects are first transferred to the boundary of the solution domain. Then the accurate solution of problems in which there are boundary singularities is demonstrated by including the analytic nature of the singular solution in only those regions nearest the singularity. Because of this, only a minor modification of the classical nonlinear BIE is required and this results in a substantial improvement in the accuracy of the numerical results throughout the entire solution domain. The BIE has previously been applied to *either* nonlinear *or* singular problems and so the method presently described constitutes an extension in this field. © 1984 Academic Press, Inc.

INTRODUCTION

Integral methods for solving boundary value problems in mathematics, engineering and physics have been gaining in popularity in recent years [1–3]. In such methods the governing field equations are recast, by application of the divergence theorem [4], into a system of coupled integral equations which apply only on the boundary of the solution domain. These integral equations are usually intractable by analytic methods and thus much attention has been given to their numerical solution using a variety of approximating techniques [5–7].

One immediate advantage of the reformulation is that the equations apply only on the boundary of the solution domain whereas space discretisation techniques such as finite difference (FD) or finite element (FE) evaluate information at many interior points. The boundary integral equation (BIE) method uses only the boundary data to compute the solution at any interior point and it is found that a high degree of resolution may be obtained. An immediate consequence of this is that the system of algebraic equations generated by a BIE is considerably smaller than that generated by an equivalent FD or FE approximation.

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The BIE method has, over the last decade, proved to be an effective tool for the numerical solution of two-dimensional harmonic boundary value problems (BVP) containing singularities (e.g., [5, 7]). It is well known that the presence of one or more boundary singularities tends to decrease the rate of convergence of numerical solution with decreasing mesh size, a phenomenon first noted by Motz [8] and Woods [9], who investigated singular elliptic problems using FD and relaxation techniques. Symm [5] showed how the classical BIE [1] could be modified to incorporate the analytic nature of a singularity whenever it occurred on the boundary of the solution domain. The results of this technique, subsequently referred to as singularity subtraction (SS), were appreciably more accurate than those of the classical BIE, and encouraged applications in the fields of heat transfer [7], electrostatics [10] and viscous fluid mechanics [11].

The improved accuracy of the SS technique was obtained at the expense of a large increase in analysis and computer code. More recently, Xanthis *et al.* [12, Sect. 3.2] suggested a method in which the analytic nature of the singularity is incorporated into the BIE by the introduction of special functional behaviour over those segments of the boundary nearest the singularity. This greatly reduces the amount of extra analysis and programming required to modify the classical BIE, and the results of this method are of comparable accuracy to the SS technique.

The solution of nonlinear problems in heat transfer using BIE techniques is well established, see, for example, the investigations of Khader [13], Bialecki and Nowak [14], Ingham *et al.* [15] and Khader and Hanna [16]. Each of the studies [13–16] deals with the aspects of nonlinear boundary conditions while [13], [14] and [16] deal with temperature-dependent thermal conductivity by first employing the Kirchoff transformation [17]. However, all of these studies were restricted to nonsingular problems.

The contribution of the present work is to combine the singular character of the solution with the application of the Kirchoff transformation to solve the full singular, nonlinear problem, i.e., nonlinear governing equation, nonlinear boundary conditions and boundary singularity. The singularity treatment in [12] was used rather than the SS technique because it is far from clear whether or not the SS technique is applicable to nonlinear integral equations, and even if it were, the extra analysis and ensuing algorithm would be so excessively complicated that it would far outweigh the advantages of its application.

The method is illustrated by an application to a problem of two-dimensional steady-state heat transfer, the conducting medium having variable thermal conductivity. The boundary conditions are chosen to illustrate the applicability of the method to (a) a dominant boundary singularity and (b) highly nonlinear boundary conditions. The resulting system of nonlinear algebraic equations were solved by application of the Newton Raphson technique [18] which provided converged solutions for the whole range of system parameters considered.

Formulation

We shall consider the solution of the nonlinear elliptic equation

$$\nabla \cdot [f(\varphi)\, \nabla\varphi] = 0 \tag{1}$$

for the two-dimensional potential φ in the region Ω enclosed by boundary $\partial\Omega$. Here f may be any function of φ such that $f(\varphi)$ remains bounded throughout $\Omega + \partial\Omega$, and so depending on the nature of f, Eq. (1) may be highly nonlinear. Note that Eq. (1) has physical applications in, for example, the field of heat transfer in which case f is the thermal conductivity of the medium within $\partial\Omega$ and φ is the temperature field [14], and in the field of magnetostatics, where f is then the magnetic permeability, φ the magnetic scalar potential and Eq. (1) reduces to the Maxwell equation $\nabla \cdot \mathbf{B} = 0$ for the magnetic field $\mathbf{B}$ [19].

The first step in solving Eq. (1) using a BIE formulation is the introduction of the transformed variable T which satisfies

$$\nabla T = f(\varphi)\, \nabla\varphi. \tag{2}$$

Equation (2) is a form of the Kirchoff transformation given in [13, 14, 16], and may be justified by noting that the curl of the right-hand side is identically zero for any functions f and φ. Then from Eqs. (1) and (2) T satisfies Laplace's equation

$$\nabla^2 T = 0 \tag{3}$$

in $\Omega + \partial\Omega$. The application of the BIE method to the solution of Eq. (3) is well documented [5–7, 15] and consequently only those points necessary to facilitate concise explanation of the present method will be included.

Green's Integral Formula for T may be expressed as (see Ref. [1])

$$\eta(p)\, T(p) = \int_{\partial\Omega} \{T(q) \log' |p - q| - T'(q) \log |p - q|\}\, dq \tag{4}$$

where (i) $p \in \Omega + \partial\Omega$, $q \in \partial\Omega$; (ii) dq denotes the differential increment of $\partial\Omega$ at q; (iii) a prime denotes differentiation with respect to the outward normal to $\partial\Omega$ at q; (iv) $\eta(p)$ is defined by

$$\eta(p) = \begin{cases} 0 & \text{if } p \notin \Omega + \partial\Omega \\ \alpha & \text{if } p \in \partial\Omega \\ 2\pi & \text{if } p \in \Omega \end{cases}$$

where α is the angle included between the tangents to $\partial\Omega$ on either side of p. If either T or T' are prescribed at each point $q \in \partial\Omega$ then the solution of the boundary integral equation obtained by letting $p = q \in \partial\Omega$ in Eq. (4) determines the boundary distribution of both T and T'. Equation (4) may now be used to generate the solution $T(p)$ at any point $p \in \Omega + \partial\Omega$.

Defining $g(\varphi)$ by

$$g(\varphi) = \varphi^{-1} \int^{\varphi} f(\beta)\, d\beta \tag{5}$$

and employing Eq. (2), we may write the Kirchoff transformation in the form

$$T = \varphi g(\varphi), \qquad T' = \varphi' f(\varphi) \tag{6}$$

relating the original and transformed BIE variables. Combining Eqs. (4) and (6) then gives

$$\begin{aligned}\int_{\partial\Omega} &\{\varphi(q)\, g[\varphi(q)] \log' |p-q| - \varphi'(q) f[\varphi(q)] \log |p-q|\}\, dq \\ &- \eta(p)\, \varphi(p)\, g[\varphi(p)] = 0, \qquad p \in \Omega + \partial\Omega, q \in \partial\Omega\end{aligned} \tag{7}$$

as the nonlinear integral equation on $\partial\Omega$. Iterative solution of this equation (plus prescribed boundary conditions) constitutes the BIE solution to this problem (e.g., [15]).

Although the formulation is applicable to problems containing a general bounded function $f(\varphi)$, we shall restrict our study to a physical problem in heat transfer. In this case the function f is the thermal conductivity of the medium and is denoted by k. For any medium, k is usually obtained on the basis of experimental results which provide some form of empirical relationship with φ [20]. It is found that k usually exhibits an almost-linear variation with φ and deviates from this behaviour only at large temperatures. The present formulation permits any bounded variation of k with φ but we shall illustrate the method with a particular example in which k is a polynomial in φ. For simplicity we shall assume a quadratic variation, namely,

$$k(\varphi) = k_0 + k_1\varphi + k_2\varphi^2 \tag{8}$$

in which there are no restrictions on the values of the real constants k_0, k_1 and k_2.

By considering the ways in which heat may be dissipated across the surface of a body into the surrounding medium one finds that there is a contribution from convection, which obeys Newton's law of cooling [20]

$$\varphi'_{\text{conv}} \propto \varphi - \varphi_{\text{amb}} \tag{9a}$$

and another from radiation, which obeys Stefan's law [20]

$$\varphi'_{\text{rad}} \propto (\varphi - \varphi_{\text{amb}})^4 \tag{9b}$$

where φ_{amb} is the temperature of the region surrounding $\partial\Omega$. Adding to these the possibility of heat sources or sinks on $\partial\Omega$, the most general form of the boundary heat flux condition is therefore

$$\varphi'(q) = \alpha(q) + \beta(q)\,\varphi(q) + \gamma(q)[\varphi(q)]^4, \qquad q \in \partial\Omega \tag{10}$$

where $\varphi' \equiv \partial\varphi/\partial n$ is the heat flux across $\partial\Omega$ and the scaling is such that φ_{amb} is zero. Thus the coefficients α, β and γ in Eq. (10) are, respectively, those related to the heat source/sink, convection and radiation at each point $q \in \partial\Omega$.

We choose Ω to be the rectangle defined by $-5 \leqslant x \leqslant 5$, $0 \leqslant y \leqslant 1$ so that certain results may be compared with those of Symm [5] and Whiteman and Papamichael [21]. Further, we prescribe the boundary conditions on $\partial\Omega$ to be

$$\varphi = 1 \qquad \text{on} \quad y = 0, x < 0 \tag{11a}$$

$$\varphi' = 0 \qquad \text{on} \quad y = 0, x > 0 \tag{11b}$$

$$\varphi' = \alpha_1 + \beta_1\varphi + \gamma_1\varphi^4 \qquad \text{on} \quad x = 5 \tag{11c}$$

$$\varphi = 0 \qquad \text{on} \quad y = 5 \tag{11d}$$

$$\varphi' = \alpha_2 + \beta_2\varphi + \gamma_2\varphi^4 \qquad \text{on} \quad x = -5 \tag{11e}$$

so that there is a discontinuity in boundary conditions on $y = 0$ at $x = 0$ which we shall subsequently refer to as the singularity S (see Fig. 1). The parameters k_0, k_1, k_2, α_1, β_1, γ_1, α_2, β_2 and γ_2 of Eq. (8) and conditions (11) are real constants, and conditions (11) are such that when $k_0 = 1$ and the other eight parameters are all zero, the problem is linear so that results may be compared with the SS BIE of Symm [5], as well as the analytic solution generated by the conformal transformation method (CTM) of Whiteman and Papamichael [21]. This allows a comparison of our results with those of alternative analytical and numerical schemes.

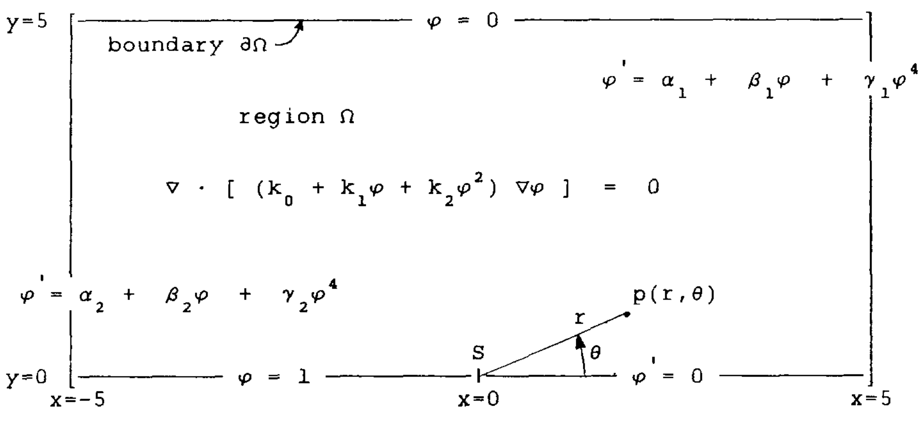

FIG. 1. Solution domain and boundary conditions.

The method used for the treatment of the singularity is a basic variant of that presented by Xanthis *et al.* [12, Section 3.2] in which the solution near S is approximated by the introduction of special functions displaying the required singular behaviour. The methods in [12] give a more sophisticated treatment of the solution in the remainder of Ω than does the present work, but we shall show that sufficiently accurate results were obtained using the present formulation. We begin by using Eq. (6) to transform the boundary conditions (11a) and (11b) so that

$$T = g(1) = T_0, \qquad \text{say, on } y = 0, x < 0 \tag{12a}$$

$$T' = 0 \qquad \text{on } y = 0, x > 0. \tag{12b}$$

By considering separated solutions of Eq. (3) in plane polar coordinates (r, θ) centered on S (see Fig. 1), and enforcing conditions (12a) and (12b), we find that [5, 8]

$$\begin{aligned} T(r, \theta) &= T_0 + \varepsilon_1 r^{\lambda_1} \cos \lambda_1 \theta + \varepsilon_2 r^{\lambda_2} \cos \lambda_2 \theta + \cdots \\ \lambda_n &= (2n-1)/2, \qquad n = 1, 2, \ldots \end{aligned} \tag{13}$$

in the neighbourhood of S. The constants ε_n, $n = 1, 2, \ldots$, are referred to as the singularity expansion coefficients. Observe that Eq. (13) shows that S is an "$r^{1/2}$ singularity" and is therefore the most dominant form of singularity possible for harmonic problems. Hence if the present method is effective on this form of singularity it should readily cope with weaker forms of singular behaviour.

In practice Eqs. (4) and (7) may rarely be solved analytically and so a numerical solution procedure is adopted [5]. The boundary is first discretised into N straight line segments $\partial\Omega_j$, $j = 1, \ldots, N$, on each of which the variables T and T' are approximated by the piecewise constant values T_j and T'_j. This is the "classical" BIE for the solution of Eq. (4) [5, 7]. In the present formulation the behaviour of Eq. (13) is incorporated into the boundary variation in T and T' on the boundary segments $\partial\Omega_j$ nearest S. By enforcing this behaviour on, for example, the M segments nearest S, we are able to evaluate the constants $\varepsilon_1, \ldots, \varepsilon_M$ since the unknown physical variables T and T' are replaced by linear combinations of the unknown ε's on these segments. This means that if the behaviour of Eq. (13) is enforced at M segments nearest S, it must be in the form of a truncated series terminating in the ε_M term.

Numbering the boundary segments in an anticlockwise direction from S so that S represents the common endpoint of segments $\partial\Omega_1$ and $\partial\Omega_N$, we now postulate that the behaviour of Eq. (13) applies on $\partial\Omega_1$ and $\partial\Omega_N$, so that for the present illustration, we require only the values of ε_1 and ε_2. Assuming piecewise constancy of T and T' over the remaining segments, we therefore have

$$T = T_j, \qquad T' = T'_j \qquad \text{on} \quad \partial\Omega_j, j = 2, \ldots, N-1 \tag{14a}$$

$$T = T_0 + \varepsilon_1 r^{1/2} + \varepsilon_2 r^{3/2} \qquad \text{on} \quad \partial\Omega_1 \tag{14b}$$

$$T' = -\tfrac{1}{2}\varepsilon_1 r^{-1/2} + \tfrac{3}{2}\varepsilon_2 r^{1/2} \qquad \text{on } \partial\Omega_N. \tag{14c}$$

On the basis of approximations (14), the discretised form of Eq. (4) becomes

$$\sum_{j=2}^{N} T_j \int_{\partial\Omega_j} \log' |p-q|\, dq - \sum_{j=1}^{N-1} T_j' \int_{\partial\Omega_N} \log |p-q|\, dq$$
$$+ \int_{\partial\Omega_1} (T_0 + \varepsilon_1 r^{1/2} + \varepsilon_2 r^{3/2}) \log' |p-q|\, dq$$
$$- \int_{\partial\Omega_N} (-\tfrac{1}{2}\varepsilon_1 r^{-1/2} + \tfrac{3}{2}\varepsilon_2 r^{1/2}) \log |p-q|\, dq$$
$$- \eta(p)\, T(p) = 0 \tag{15}$$

where $p \in \Omega + \partial\Omega$, $q \in \partial\Omega$ and $r = r(q)$. Collocating Eq. (15) at the midpoint $p \equiv q_i$, $i = 1,\ldots, N$, of each boundary segment generates the system of algebraic equations

$$\sum_{j=2}^{N} A_{ij} T_j + \sum_{j=1}^{N-1} B_{ij} T_j' + A_{i1} T_0 + \varepsilon_1 G_i + \varepsilon_2 H_i = 0, \qquad i = 1,\ldots, N \tag{16}$$

where

$$A_{ij} = \int_{\partial\Omega_j} \log' |q_i - q|\, dq - \delta_{ij}\eta(q_i) \tag{17}$$

$$B_{ij} = -\int_{\partial\Omega_j} \log |q_i - q|\, dq \tag{18}$$

and

$$G_i = C_{i1} - \tfrac{1}{2}E_{iN} - \eta(q_1)\, r^{1/2}(q_1)\, \delta_{i1} \tag{19}$$

$$H_i = D_{i1} + \tfrac{3}{2}F_{iN} - \eta(q_1)\, r^{3/2}(q_1)\, \delta_{i1} \tag{20}$$

where

$$C_{i1} = \int_{\partial\Omega_1} r^{1/2}(q) \log' |q_i - q|\, dq \tag{21}$$

$$D_{i1} = \int_{\partial\Omega_1} r^{3/2}(q) \log' |q_i - q|\, dq \tag{22}$$

$$E_{iN} = -\int_{\partial\Omega_N} r^{-1/2}(q) \log |q_i - q|\, dq \tag{23}$$

$$F_{iN} = -\int_{\partial\Omega_N} r^{1/2}(q) \log |q_i - q|\, dq. \tag{24}$$

In each of the expressions (19)–(24), $r(q) = |q(S) - q|$ where $q(S)$ is the position of the singularity, and δ_{ij} is the Kronecker delta. The integrals A_{ij} and B_{ij} may be evaluated analytically regardless of the position of q_i provided that $\partial\Omega_j$ is a straight line segment [5, 7]. However, the integrals C_{i1}, D_{i1}, E_{iN} and F_{iN} may only be evaluated analytically when q_i is collinear with the straight line segments $\partial\Omega_1$ and $\partial\Omega_N$: the resulting expressions are given in the Appendix. For the general position of q_i, C_{i1}, D_{i1}, E_{iN} and F_{iN} were evaluated numerically to a relative error of 10^{-8} using Patterson's Quadrature [22].

Reverting to the physical variables, Eqs. (7) and (16) give the algebraic equations for φ and φ' as

$$\sum_{j=2}^{N} A_{ij}\varphi_j g(\varphi_j) + \sum_{j=1}^{N-1} B_{ij}\varphi'_j k(\varphi_j) + A_{i1}T_0$$
$$+ \varepsilon_1 G_i + \varepsilon_2 H_i = 0, \qquad i = 1,\ldots, N. \tag{25}$$

These nonlinear algebraic equations are supplemented by Eq. (8) and the discretised forms of conditions (12). The complete system lends itself ideally to solution by the Newton Raphson method [18], as g and k are known functional forms of φ and therefore the Jacobian of the system may readily be evaluated by explicit partial differentiation. The iterative procedure employed is similar to that used by Ingham *et al.* [15] and so the details are not reproduced here. As noted by Ingham *et al.* [15], the convergence of such an iterative procedure is not necessarily guaranteed even if the iteration is initiated with a guess close to the desired root. However, no such difficulties were encountered in the present work: a wide range of system parameters were tested and convergence was always achieved even for the most nonlinear systems. Writing the unknowns of Eqs. (25) as the N-vector

$$\mathbf{x} = (\varepsilon_1, \varphi_2, \varphi_3, \ldots, \varphi'_{N-2}, \varphi'_{N-1}, \varepsilon_2)^T \tag{26}$$

and hence rewriting Eqs. (25) as

$$F_i(\mathbf{x}) = 0, \qquad i = 1,\ldots, N \tag{27}$$

then convergence was considered achieved when both

$$|F_i(\mathbf{x}^{(n)})| < 10^{-8}, \qquad i = 1,\ldots, N \tag{28}$$

and

$$\left| \frac{x_i^{(n)} - x_i^{(n+1)}}{x_i^{(n)}} \right| < 10^{-8}, \qquad i = 1,\ldots, N \tag{29}$$

where a superscript n refers to the value of the unknowns on the nth iteration. The iterations were initiated with

$$x_i^{(1)} = 1, \qquad i = 1,\ldots, N.$$

Having solved for $\mathbf{x}$, the values of ε_1, ε_2; $\varphi_2,\ldots,\varphi_N$ and $\varphi'_1,\ldots,\varphi'_{N-1}$ are all known. Equations (6) are then used to evaluate $T_2,\ldots,T_N$ and $T'_1,\ldots,T'_{N-1}$. Then a discretised form of Eq. (15) is used to solve for $T(p)$ at the general point $p \in \Omega + \partial\Omega$. Finally, $\varphi(p)$ is obtained from the first of Eqs. (6) using Newton Raphson. The scaling in the formulation is such that $0 \leqslant \varphi \leqslant 1$ and in each of the problems considered in the present work, it was found that the $\varphi(p)$ so obtained was the unique solution within the above interval. Hence we have been able to evaluate the unique real solution even in the cases when the governing equation and boundary conditions are highly nonlinear.

It is a straightforward process to improve the accuracy still further by assuming a piecewise linear or piecewise quadratic [6, 7] variation of T and T' in (14a). However, this constitutes a change in the details rather than the concepts of the present work. In fact, using only a piecewise constant variation of T and T' on $\partial\Omega$, the results were found to converge extremely rapidly with decreasing mesh size.

Results and discussion

The large number of system parameters precludes any exhaustive variation in the parameters of Eq. (8) and conditions (11). Consequently results were obtained for five different parameter lists, referred to as cases I, II, III, IV and V having the following details:

Case	k_0	k_1	k_2	α_1	β_1	γ_1	α_2	β_2	γ_2
I	1.0	0	0	0	0	0	0	0	0
II	1.0	0.1	−0.01	0	−0.01	0	0	−0.1	−0.01
III	1.0	1.0	1.0	0	−0.01	0	0	−0.1	0
IV	1.0	1.0	1.0	0	−1.0	−1.0	0	−1.0	−1.0
V	1.0	10.0	100.0	0	−1.0	−1.0	0	−10.0	−10.0

Case I represents the linear problem whose results will be compared with those of the SS technique [5] and the CTM [21]. Case II is one in which the system parameters are physically appropriate to many heat transfer problems. The nonlinearity of the problem increases in terms of the boundary conditions as we go from case III to case IV and then in terms of the coefficients of the governing equation as we go from case IV to case V, so that case V represents an extremely nonlinear problem.

In Table I we present the values of the coefficients in the truncated series expansion (13), denoted by ε_1 and ε_2. Results for each case are given for four discretisations comprising 30, 60, 120 and 240 segments of equal length so that the convergence of results with decreasing mesh size may be investigated. So rapid is the rate of convergence that the values of ε_1 at $N=30$ vary by less than 0.25% from those at $N=240$ for even the most nonlinear case, this variation diminishing to 0.05% by the

TABLE I

Singularity Series Expansion Coefficients

Case	N	ε_1	ε_2	N_{iter}
I	30	−0.4843	0.0313	2
I	60	−0.4844	0.0314	2
I	120	−0.4845	0.0314	2
I	240	−0.4845	0.0315	2
II	30	−0.5117	0.0343	4
II	60	−0.5117	0.0344	4
II	120	−0.5118	0.0345	4
II	240	−0.5119	0.0345	4
III	30	−0.8983	0.0608	5
III	60	−0.8984	0.0609	5
III	120	−0.8985	0.0610	6
III	240	−0.8986	0.0611	6
IV	30	−0.9768	0.0634	6
IV	60	−0.9758	0.0639	6
IV	120	−0.9757	0.0640	7
IV	240	−0.9758	0.0641	6
V	30	−21.4208	1.4801	9
V	60	−21.3866	1.4902	9
V	120	−21.3791	1.4912	9
V	240	−21.3780	1.4914	9

time $N = 60$. The results of case I may be compared with those of Symm [5], who obtains the values $\varepsilon_1 = -0.4835$ and $\varepsilon_2 = 0.02988$, so that the ε_1 of the present method is only 0.2% in error from that predicted by the SS technique. However, the value of ε_2 is some 3% in error from that obtained in [5] and this is due to the fact that in the SS technique, the behaviour of Eq. (13) is applied to the whole of Ω, whereas in the present study it is restricted to the region $\{r: 0 \leqslant r \leqslant O(N^{-1})\}$. Thus the present modification of the BIE gives comparable convergence to the SS technique even though it requires far less programming and code.

Also given in Table I are the number of iterations required for the satisfaction of conditions (28) and (29), where for each case and discretisation, all unknowns in the iterative scheme were initially set equal to unity. Note that even the most nonlinear problem, case V, required as few as nine iterations to satisfy the convergence criteria.

Table II shows classical BIE results of $\varphi(p)$ for case V at equally spaced field points in the solution domain $-5 \leqslant x \leqslant 5$, $0 \leqslant y \leqslant 1$ for the discretisations $N = 30$,

TABLE II

Classical BIE Results for Case V

N = 30
= 60
= 120
= 240

-0.2670	0.0304	0.0220	0.0223	0.0130	0.0115	0.0047	-0.0002	-0.0023	-0.0002	-0.1504
-0.1428	0.0019	0.0031	0.0031	0.0024	0.0016	0.0007	0.0000	-0.0003	-0.0002	-0.0731
-0.0417	0.0003	0.0004	0.0004	0.0003	0.0002	0.0001	0.0000	0.0000	0.0000	-0.0200
-0.0099	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-0.0067
0.0849	0.3468	0.4301	0.4653	0.4709	0.4550	0.4230	0.3789	0.3247	0.2563	0.1507
0.0863	0.3477	0.4311	0.4668	0.4730	0.4579	0.4264	0.3826	0.3282	0.2596	0.1532
0.0874	0.3480	0.4316	0.4675	0.4741	0.4593	0.4281	0.3844	0.3300	0.2612	0.1544
0.0876	0.3482	0.4319	0.4679	0.4747	0.4600	0.4290	0.3853	0.3309	0.2620	0.1550
0.1130	0.4781	0.5815	0.6219	0.6250	0.6005	0.5552	0.4960	0.4257	0.3394	0.2041
0.1293	0.4792	0.5826	0.6236	0.6276	0.6043	0.5599	0.5008	0.4303	0.3434	0.2063
0.1312	0.4796	0.5832	0.6244	0.6290	0.6062	0.5622	0.5032	0.4326	0.3454	0.2075
0.1315	0.4797	0.5834	0.6248	0.6297	0.6072	0.5634	0.5045	0.4338	0.3464	0.2082
0.1603	0.6054	0.7166	0.7535	0.7502	0.7131	0.6499	0.5736	0.4894	0.3899	0.2355
0.1712	0.6066	0.7174	0.7549	0.7531	0.7181	0.6560	0.5796	0.4948	0.3945	0.2373
0.1756	0.6070	0.7178	0.7557	0.7546	0.7206	0.6591	0.5827	0.4976	0.3969	0.2392
0.1762	0.6071	0.7180	0.7561	0.7553	0.7218	0.6606	0.5843	0.4990	0.3981	0.2399
0.2304	0.7713	0.8585	0.8806	0.8719	0.8161	0.7211	0.6241	0.5276	0.4150	0.2523
0.2319	0.7714	0.8585	0.8813	0.8744	0.8238	0.7294	0.6313	0.5338	0.4242	0.2556
0.2428	0.7715	0.8587	0.8818	0.8756	0.8274	0.7336	0.6350	0.5370	0.4268	0.2572
0.2455	0.7716	0.8588	0.8820	0.8763	0.8291	0.7357	0.6369	0.5366	0.4282	0.2561
0.8036	1.0305	1.0002	1.0006	1.0084	0.9232	0.7527	0.6420	0.5399	0.4272	0.2416
0.8147	1.0001	1.0002	1.0001	1.0000	0.9456	0.7630	0.6502	0.5470	0.4336	0.2544
0.8345	1.0002	1.0000	1.0000	1.0000	0.9620	0.7685	0.6544	0.5505	0.4367	0.2604
0.8611	1.0000	1.0000	1.0000	1.0000	0.9733	0.7713	0.6565	0.5523	0.4382	0.2633

S

60, 120 and 240. Inspection of the results near the singularity S reveals the poor convergence of results with increasing N. Moreover, the results throughout the entire solution domain are somewhat slow to converge as the mesh is refined.

In Table III we present an equivalent distribution of results for case V obtained with the present method. The rate of convergence has been dramatically accelerated so that for $N = 60$, results at interior points differ by only $O(0.01\%)$ from those obtained with $N = 240$, as opposed to $O(1\%)$ for the equivalent classical BIE discretisation. The only region of Ω at which this level of convergence has not quite been achieved is along the boundary $x = -5$, where inspection of condition (11e) reveals that for case V, there are highly nonlinear contributions to the boundary condition from the radiative flux term.

The accuracy quoted above held throughout the entire solution domain in case I to IV—only the results of case V have been presented in order to demonstrate the fact that convergence was obtained for even the most nonlinear system.

The choice of parameters in case I were such that a comparison with the analytical results of the CTM [21] could be made. It was found that the potential evaluated at the general field point using the present BIE differed from that in [21] by only $O(0.01\%)$ for N as low as 60 thus indicating the rapid acceleration in the rate of convergence of BIE results to the exact analytical results.

TABLE III

Modified BIE Results for Case V

N = 30
= 60
= 120
= 240

-0.2676	0.0334	0.0218	0.0221	0.0160	0.0121	0.0056	0.0005	-0.0020	0.0000	-0.1652
-0.1430	0.0019	0.0031	0.0030	0.0025	0.0016	0.0007	0.0001	-0.0003	-0.0002	-0.0749
-0.0417	0.0003	0.0004	0.0004	0.0003	0.0002	0.0001	0.0000	0.0000	0.0000	-0.0202
-0.0099	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-0.0067
0.0853	0.3478	0.4317	0.4678	0.4747	0.4604	0.4296	0.3863	0.3319	0.2626	0.1547
0.0865	0.3482	0.4320	0.4681	0.4750	0.4606	0.4297	0.3862	0.3318	0.2627	0.1552
0.0875	0.3483	0.4321	0.4682	0.4752	0.4607	0.4298	0.3862	0.3318	0.2627	0.1554
0.0876	0.3484	0.4321	0.4683	0.4752	0.4608	0.4298	0.3862	0.3318	0.2628	0.1555
0.1135	0.4791	0.5832	0.6247	0.6296	0.6075	0.5643	0.5059	0.4352	0.3475	0.2092
0.1294	0.4797	0.5835	0.6251	0.6301	0.6079	0.5644	0.5057	0.4349	0.3473	0.2085
0.1313	0.4799	0.5837	0.6252	0.6303	0.6081	0.5645	0.5057	0.4349	0.3473	0.2085
0.1316	0.4799	0.5837	0.6253	0.6303	0.6081	0.5645	0.5057	0.4349	0.3474	0.2086
0.1607	0.6061	0.7179	0.7559	0.7549	0.7218	0.6622	0.5865	0.5009	0.3994	0.2414
0.1719	0.6071	0.7181	0.7563	0.7557	0.7227	0.6621	0.5858	0.5004	0.3991	0.2407
0.1756	0.6072	0.7182	0.7564	0.7560	0.7229	0.6621	0.5858	0.5004	0.3992	0.2406
0.1762	0.6072	0.7182	0.7565	0.7560	0.7230	0.6621	0.5858	0.5004	0.3992	0.2407
0.2304	0.7716	0.8591	0.8819	0.8753	0.8274	0.7399	0.6404	0.5407	0.4294	0.2586
0.2320	0.7716	0.8589	0.8821	0.8765	0.8301	0.7382	0.6389	0.5402	0.4293	0.2588
0.2428	0.7716	0.8589	0.8822	0.8766	0.8307	0.7379	0.6388	0.5402	0.4294	0.2586
0.2455	0.7716	0.8589	0.8823	0.8769	0.8308	0.7378	0.6388	0.5402	0.4294	0.2589
0.8036	1.0305	1.0002	1.0011	0.9998	1.0004	0.7740	0.6582	0.5531	0.4376	0.2472
0.8147	1.0001	1.0002	1.0001	1.0005	1.0002	0.7740	0.6584	0.5537	0.4390	0.2576
0.3345	1.0002	1.0000	1.0000	1.0000	1.0001	0.7740	0.6585	0.5539	0.4394	0.262[illegible]
0.8611	1.0000	1.0000	1.0000	1.0000	1.0001	0.7741	0.6586	0.5539	0.4396	0.2637

S

Conclusions

A modification of the classical BIE has been presented which enables accurate treatment of a class of nonlinear elliptic equations containing boundary singularities. The method requires a slight modification of the classical BIE with the reward of a dramatic improvement in the rate of convergence of results throughout the entire solution domain. It may be applied to any geometry for which the analytic form of the singularity can be obtained and the use of the Kirchoff transformation means that it is applicable to problems posed in terms of T which could not be treated by the SS technique were the problem to be posed in terms of φ.

The form of the singularity will of course affect the integrals C_{i1}, D_{i1}, E_{iN} and F_{iN} but these may still be evaluated with sufficient accuracy via Patterson's Quadrature [22] and so impose no restriction on the class of problems to which the method may be applied.

The present paper has concentrated on the example in heat transfer but the method is equally well suited to solving any equation of the form

$$\nabla \cdot [f(\varphi)\, \nabla(\varphi)] = 0$$

for which the (bounded) dependence of f on φ is known.

As the analytic nature of the singularity is incorporated only on those boundary segments nearest S, the modification of the classical BIE is minimal and so requires very little extra cpu time. Moreover the improved convergence properties mean that accurate results may be obtained for relatively crude discretisations.

Note that if the boundary was approximated by piecewise-curved sections on either side of the singularity, the imposition of boundary conditions (12) on the sections nearest the singularity could only be effected after one (or more) linearising transformation(s): this in turn would transform the remainder of the boundary into a more intricate geometry and would complicate the ensuing algorithm substantially. To include the solution of such a problem in the present paper would detract from the main aim of this work, which was to combine the application of the Kirchoff transformation with the treatment of singularities.

APPENDIX

The integrals C_{i1}, D_{i1}, E_{iN} and F_{iN} may be evaluated analytically when q_i and the endpoints of the straight line segment $\partial\Omega_N$ are collinear. From Fig. 2, there are only five qualitatively distinct positions of q_i relative to $\partial\Omega_N$ and these are: (i) $q_i = q_R$; (ii) $q_i = q_N$; (iii) $q_i = q_M$; (iv) $q_i = q_{N-1}$ and (v) $q_i = q_L$. Using the notation of Fig. 2, let $h = |q_{N-1} - q_N|$, $a = |q_{N-1} - q_i|$ and $b = |q_N - q_i|$. Then in each of cases (i)–(v), $C_{i1} = D_{i1} = 0$, and the remaining integrals are as follows:

(i) $q_i = q_R$: if $I_1 = 2b^{1/2}\tan^{-1}[(h/b)^{1/2}] - 2h^{1/2}$
and $I_2 = -\frac{2}{3}h^{3/2} - bI_1$
then $E_{iN} = -2[h^{1/2}\log a + I_1]$
and $F_{iN} = -\frac{2}{3}[h^{3/2}\log a + I_2]$;

(ii) $q_i = q_N$: if $I_1 = -2h^{1/2}$
and $I_2 = -\frac{2}{3}h^{3/2}$
then $E_{iN} = -2[h^{1/2}\log h + I_1]$
and $F_{iN} = -\frac{2}{3}[h^{3/2}\log h + I_2]$;

(iii) $q_i = q_M$: if $I_1 = b^{1/2}\log|(b^{1/2} + h^{1/2})/(b^{1/2} - h^{1/2})| - 2h^{1/2}$
and $I_2 = -\frac{2}{3}h^{3/2} + bI_1$
then E_{iN} and F_{iN} are as in (i);

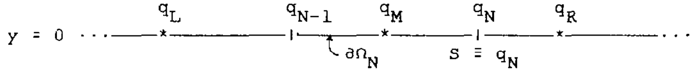

FIG. 2. Notation for analytic evaluation of integrals.

(iv) $q_i = q_{N-1}$: if $I_1 = h^{1/2} \log 4 - 2h^{1/2}$
and $I_2 = -\frac{2}{3}h^{3/2} + hI_1$
then E_{iN} and F_{iN} are as in (ii);

(v) $q_i = q_L$: both E_{iN} and F_{iN} are as in (iii).

Although extra analysis is required for the analytic evaluation of these integrals, one finds them necessary insofar as the numerical quadrature may become computationally expensive when q_i is on or near $\partial\Omega_N$, for then the kernels of the integrals E_{iN} and F_{iN} become logarithmically singular. These analytic integrations ensure accurate results in the neighbourhood of S and are of course applicable to any problem containing a singularity of the form given in (13).

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